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Linear Difference Equations: Integrated– Environmentalist Approach for Teaching Mathematics

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There are at least eight known approaches for teaching mathematics that have been described in the literature. These include the Integrated-Environmentalist, Formative, Social-Constructivist, Structuralist, New Maths, Problem Solving, Cultural, and Behaviorist approaches (Neyland, 1995). This paper presents some examples for teaching mathematics using the Integrated-Environmentalist approach, which is based on the assumption that the content knowledge in mathematics cannot be separated from the meaningful context from which it is taken, and in which it can be explained. In this approach for teaching mathematics the environment is used as the main source of mathematical meaning, stimulation to do creative work in mathematics, and is the base for the abstraction process.

Difference equations.

A difference equation is a mathematical equality involving the differences between successive values of a function of a discrete variable. Thus, a population of antelope x_n in a given year may depend on the population value in the previous year (x_{n-1}), or in functional notation, $x_n = f(x_{n-1})$. As an example, if the population doubles every year, then $x_n = 2x_{n-1}$. By introducing the student to the most general form of a first order linear difference equations (for which f is a linear function), and the second order Fibonacci difference equation, the student can see applications of algebraic ideas to problems in the life sciences. In the process, students are exposed to arithmetic and geometric sequences and series, the solution of exponential and quadratic equations, the solution of simultaneous algebraic equations, and the derivation of the formula for the n^{th} Fibonacci numbers.

Difference equations are useful in many life science related contexts such as population growth or decline and pharmokinetics (e.g. drug dosage levels) to name but two. For more information see Chapter 5 of the book by Bodine *et al*., (2014). Difference equations model population or drug concentrations, x_n, at different discrete times, n, to analyze measurements that cannot be continuously recorded. Difference equations should not be confused with *differential* equations; Difference equations are much easier to solve, and are valuable for students in algebra, especially when the students are introduced to arithmetic and geometric series, quadratic equations, and exponential equations.

In terms of mathematics teaching, this article falls into the category of Integrated-Environmentalist, that is, it is seamlessly connected to the context in which this mathematical model arises. In this paper the general solution of a linear first order difference equation is explained first, followed by real world examples.

General solution of a linear first order (inhomogeneous) difference equation.

Here we will examine the difference equation in the following way. Given the difference equation $x_{n+1} = ax_n + b$, (1) for positive integer values of n, then the solution is given by

$$x_n = \left(x_0 - \frac{b}{1-a}\right)a^n + \frac{b}{1-a}; n = 0,1,2,3,\ldots \quad (2)$$

where x_0 is the initial population, and $a > 0$ and b are constants (b can be of either sign or zero). Equation (1) connects the population at one time period to the immediately preceding one in terms of the parameters a and b for any given problem. This solution (2) may be proved using the induction method, direct substitution, or direct construction. The latter will be presented here. From equation (1), since $x_n = ax_{n-1} + b$, and $x_{n-1} = ax_{n-2} + b$, it follows that

$$x_n = a(ax_{n-2} + b) + b = a^2 x_{n-2} + (a = 1)b =$$
$$= a^2(ax_{n-3} + b) + (a + 1)b = a^3 x_{n-3} + (a^2 + a + 1)b;$$

continuing this process leads to the general result

$$x_n = a^n x_0 + (a^{n-1} + a^{n-2} + \cdots + a^2 + a + 1)b \quad (3)$$

Notice when $a \neq 1$, the sum of the geometric series,

$$a^{n-1} + a^{n-2} + \cdots + a^2 + a + 1 = \frac{1 - a^n}{1 - a}$$

results in

$$x_n = a^n x_0 + \frac{1 - a^n}{1 - a}$$

which is equivalent to equation (2). If a=1 in equation (3), then the result for the arithmetic sequence is obtained, namely $x_n = x_0 + nb$. Let's put the result from equation (2) to use in the example below. The answers are embedded in the problem and the constant term b is positive, if the population is increased by b for each time period. It is negative if the population is harvested, culled or poached.

Example 1.

A population of spiny anteaters is growing at 2% per year, the initial population is 600 (exactly!). A group of middle school students is poaching them at a rate of 5 per year. Using the governing difference equation below, we will find the answers to questions a though c. We will round to the nearest appropriate integer in each case.

$$x_{n+1} = 1.02 x_n - 5$$
$$x^n = \left(600 - \frac{(-5)}{-0.02}\right)(1.02)^n + \frac{-5}{-0.02} =$$
$$= 350(1.02)^n + 250$$

(a) How many anteaters are there after 10 years?

$x_{10} = 350\,(1.02)^{10} + 250 \approx 677.$

(b) After how many years will the population double? $1200 = 350\,(1.02)^n + 250$, so we need to solve this for n to find the number of years.

$$n = \frac{\log(\frac{19}{7})}{\log 1.02} \approx 50.4$$

Since after 50 years the population will be slightly less than 1200 according to this model, we have the option of choosing 51 years. It doesn't matter what base logarithms are used to solve this problem.

(c) What should the poaching rate be to maintain a population of 600 each year? This is essentially a "fixed point" problem, when we want to find b in equation (1), when
$x_0 = ax_0 + b$, or $x_0 = 600 = 600\,(1.02) + b$.
Solving for b, $b = 12$.

This means, in order to maintain the initial population, the poaching rate must be increased to 12 spiny anteaters per year.

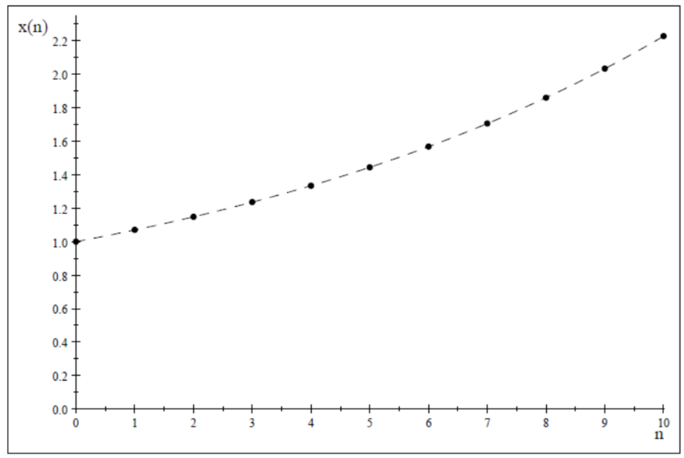

Figure 1. Exponential population growth, 12% a year

In Figure 1, the discrete population x_n ($x(n)$ in the Figure), relative to the initial population of 600) is presented for a slightly different problem: a population growth rate of 12% per year is shown instead of the 2% because the exponential-type growth of the population is easier to see – whereas for a 2% growth rate is appears to be nearly linear initially. The dotted line in the graph represents the continuous version of the solution, that is, $x_n = 350\,(1.12)^n + 250$.

Example 2: Pharmokinetics.

We will develop a difference equation mathematical model for the level D_n of medication in the bloodstream, and find the answers to the following questions:
(a) The drug dosage for a patient taking a certain statin is 10 mg/day. If the kidneys remove 60% of the drug every 24 hours, what is the maintenance level L for the medication?

If the kidneys remove a fixed proportion (1–a) of the medication from the bloodstream, then the governing difference equation is $D_{n+1} = aD_n + D_0$, if D_0 is

the repeated dosage every time period. Then with $b = D_0$ in equation (2), the solution equation is:

$$D_n = D_0\left(\frac{1-a^{n+1}}{1-a}\right)$$

When $0 < a < 1$ in problems of this kind, it is evident that

$$n \to \infty, \to D_n \to \frac{D_0}{1-a}$$

and this is called the *maintenance level*. For this problem, $L = 10 / 0.6$, which is approximately 16.7 mg.

(b) If the daily dosage was halved (or doubled), is the maintenance level halved (or doubled)? The answer is yes to both questions.

(c) Suppose the patient decides to double the dosage period and double the dosage as a way to compensate for less frequent dosage. How does this affect the maintenance level, if at all? Since 40% of the drug remains in the body after 24 hours, only 16% $[(0.4)^2 = 0.16]$ remains after 48 hours. This implies, $1-a=0.84$ and $L = 20/0.84$, which is approximately 23.8 mg. This could be very dangerous for the patient.

In Figure 2, the level of medication D_n, (D (n) in the Figure 2) is shown. As in the previous example the curved dotted line represents the continuous version of the solution. The horizontal asymptote (dotted line) represents the maintenance level $L \approx 16.7$ mg.

Phyllotaxis and the Fibonacci Difference Equation

Phyllotaxis is the distribution or arrangement of leaves on a stem and the mechanisms which govern it. The term is used by botanists and mathematicians to describe the repetitive arrangement of more than just leaves; petals, seeds, florets and branches also

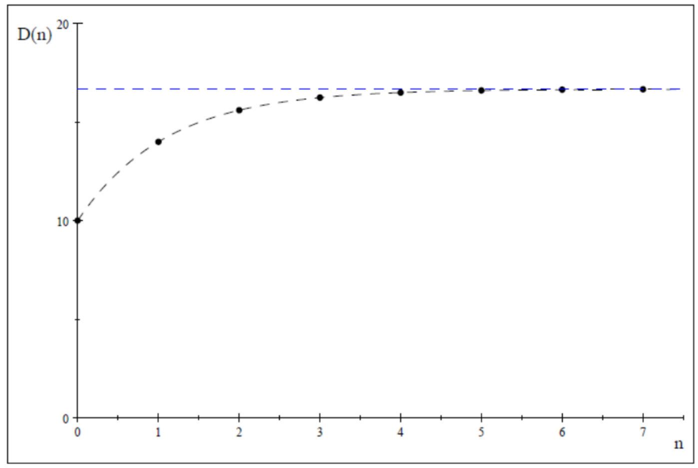

Figure 2. Level of medication, *D(n)*, with maintenance lev. *L*

have repetitive arrangements. These arrangements are related to the *Fibonacci* number sequence, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,... and to the golden number or ratio $\Phi = (1+\sqrt{5})/2 \approx 1.618034$. Sometimes, the reciprocal of Φ, Φ^{-1}, which is approximately equal to 0.618034, is referred to as the golden ratio. Numerical and geometric patterns based on these numbers abound in nature, and have been studied for hundreds of years. For that reason, the basic features of phyllotaxis are found in many elementary mathematics or science textbooks. The Fibonacci sequence is derived from a *second order* linear difference equation. This means the subscripts differ by two instead of one shown in the previous first order example. Instead of using x_n, we use F_n for the n^{th} term in the sequence. Thus, given the Fibonacci difference equation $F_{n+2} = F_{n+1} + F_n$, $n = 1, 2, \ldots$, and with $F_1 = 1$, and $F_2 = 1$, the sequence generated is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, … The question naturally arises, how do we compute a particular number such as the 30th Fibonacci number? There is an explicit formula, called the *Binet formula*, which is easily derived using the quadratic equation formula and algebra. In the derivation below the above sequence will be amended to start with zero, which satisfies the difference equation requirement for ease of computation.

We return temporarily to the solution (2) for the first order difference equation. In the equation, when $b = 0$, the corresponding solution is of the form, $F_n = Aa^n$, where A is a constant. If we place this into $F_{n+2}=F_{n+1}+F_n$ we obtain $Aa^n(a^2-a-1)=0$ which implies $a^2-a-1=0$. This implies,

$$a_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

where $a_1 \approx 1.618$ and $a_2 \approx -0.618$. There are two distinct roots for a, so we seek the general solution for F_n, when $F_n = Aa_1^n + Ba_2^n$. Thus, when $n = 0$, $Aa_1^0 + Ba_2^0 = 0$ which implies $A + B = 0$. When $n = 1$, $Aa_1^1 + Ba_2^1 = 1$ which implies $Aa_1 + Ba_2 = 1$. This implies

$$A = -B = \frac{1}{\sqrt{5}},$$

which gives us

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

for large enough *n.* The sequence effectively grows exponentially. Thus, to answer our original questions, we substitute 30 in for *n* and solve.

$$F_{30} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{30} - \left(\frac{1-\sqrt{5}}{2}\right)^{30}\right] =$$

$$= 8.3204 \times 10^5 = 832{,}040$$

Notice that the term in the second parenthesis is approximately -0.618, which rapidly diminishes in magnitude as *n* increases. It is left to the reader to find the 95^{th} Fibonacci number.

In Figure 3 the first ten Fibonacci numbers are represented graphically as *F(n)*, along with the approximation of

$$F_n \approx \frac{1}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

shown by the dotted curve. As can be seen, this is a very accurate approximation even for a small integer value for *n*.

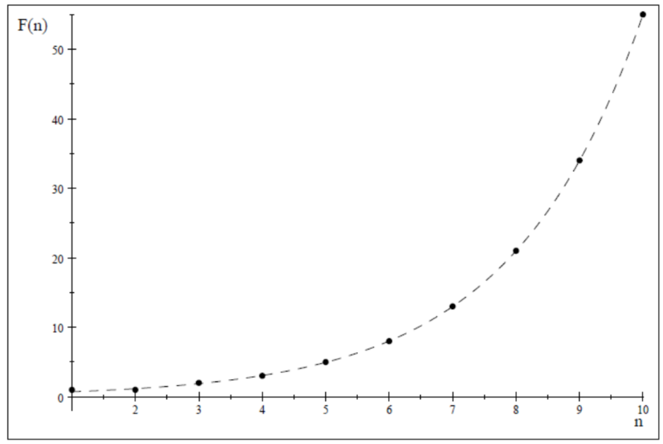

Figure 3. Graph of the Fibonacci sequence

Connection between the Fibonacci & Phyllotaxis?

We can examine some flower petals as examples of this phenomenon. Lilies have 3 petals , buttercups have 5, some delphiniums have 8 petals, marigolds have 13, asters have 21, and daisies have 34, 55 or even 89 petals. Exceptions do occur quite often: as explained by the geometer H.S.M. Coxeter, in his book, *Introduction to Geometry*. He states "... phyllotaxis is really not a universal law but only a fascinatingly prevalent tendency." Plants in general face predicaments shared by humans such as how to occupy space, receive sunlight and interact with the environment in an optimal fashion. As a branch on a plant grows upward, it produces leaves at regular angular intervals which branch out from the stem. If these angular intervals are exact rational multiples of 360°, then the leaves will grow directly above one another in a set of rays, when viewed from above. This pattern shades lower leaves from sunlight and moisture to some extent. Consider the shading patterns when leaves sprout at every 180° (1/2 revolution) or 90° (1/4 revolution). In many cases, plants use rational approximations to the golden number Φ in order to optimize leaf arrangement. In fact, depending on the plant, leaves may be generated after approximately 2/5 of a revolution of a circle, (for oak, cherry, apple, holly and plum trees), 1/2 (elm, some grasses, lime, linden), 1/3 (beech, hazel), 3/8 (poplar, rose, pear, willow) and 5/13 (almond). Other approximations include 3/5, 8/13, 5/13…, which are called *phyllotactic ratios*, and their numerators and denominators are each Fibonacci numbers, although not always consecutive ones. Many other examples are described in the book by Garland, *Fascinating Fibonaccis.* For more mathematical aspects of phyllotaxis with a connection to the Euclidean algorithm, consult Naylor (2002).

Conclusion

Introducing students to the most general form of first order linear difference equations, and the second order Fibonacci difference equation, allows them to see applications to the life sciences. Mathematically, students are exposed to arithmetic and geometric sequences and series, the solution of exponential and quadratic equations, the solution of simultaneous algebraic equations that leads to the derivation of Binet's formula for the n^{th} Fibonacci numbers. As a result, students have a mathematically rich experience that is grounded in their environment.

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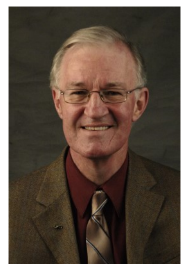

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