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Reach for the Stars!

Dimensional Analysis: Physical Insight Gained Through Algebra

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"Adam, astronomers, even observational ones, need to know an awful lot of mathematics, and you are very near the bottom of the class in algebra," said Mr. Archibald Chanter, an Algebra teacher at Henley-on-Thames Grammar School, circa 1962 (Adam, 2006).

"Algebra has often been referred to as a 'gatekeeper' to higher learning—both in mathematics and in other fields. Research shows that students who complete a mathematics course beyond the level of Algebra 2 are more than twice as likely to pursue and complete a postsecondary degree. Students who don't do well in algebra compromise their career options, especially in STEM fields" (Gojak, 2013).

In this article, I present an important topic in algebra – the solution of a system of linear equations – in the context of science: more specifically, physics, astronomy and biology (see Pennycuick (1992) for more details in the latter subject area). By solving such systems in two or three variables, students can develop physical intuition by asking *what do the solutions mean* in the context of the topic. To do this, the concept of everyday physical dimensions – speed, length, mass, time, acceleration, etc. – is reduced to combinations of fundamental 'units,' namely mass [M], length [L], and time [T]. This enables students to relate mathematical consistency in equations with dimensional consistency, and in so doing enables students to see profound connections between the mathematics and the context to which it applies.

When we write down an equation, for example $y = f(x)$, what are we really claiming? Obviously, we mean that the left-hand side and right-hand side are equal; otherwise, it is not an equation. However, when such an equation appears in a context, there is more to say. The physical dimensions of both sides must also be equal. For example, we are all familiar with the speed $v(t)$ of an object at time t, moving in a straight line (rectilinear motion) as expressed in terms of its initial speed u, constant acceleration a, and time: $v = u + at$, as a linear function of time. Regardless of the units used – miles per hour, millimeters per year, kilometers per second (or even furlongs per fortnight!) – the *dimensions* of each side are equal. To see this, note that the speeds u and v have dimensions of *length/time*. Acceleration has dimensions *length/*(*time*)^2, so the second term on the right, at, has dimensions of *length/*(*time*)^2 × *time*. This is just *length/time*, the definition of speed, as it should be. The equation is dimensionally consistent. In fact, this is a good way to check whether the equation is stated correctly; if the dimensions are inconsistent, the equation is most certainly wrong! Of course, it could be wrong for other reasons, even if the dimensions *are* consistent.

This idea of dimensional consistency is facilitated by introducing notation for three fundamental physical quantities: mass [M], length [L], and time [T]. The set {[M], [L], [T]} can be thought of like atoms from which elements can be made, or unit vectors from which all vectors in a three-dimensional space can be constructed. Thus, as above, the dimensions for acceleration, *length/* (*time*)^2, can be written as [L][T]$^{-2}$. We shall illustrate this technique that is called dimensional analysis by means of an idealized model of namely a *simple pendulum*.

A simple pendulum is defined as a point of mass m at one end of an inextensible massless string (or rigid rod) of length l; this is fixed at the other end. The simple pendulum undergoes small oscillations in the presence of a local constant gravitational acceleration g in which the friction at the connection point and the air resistance are neglected. Clearly, this is a highly idealized mathematical model! What we wish to do is find out how the period P of the pendulum (i.e. the time for it to swing from one extreme to the other and back again – "Tic-Toc"!) depends on its mass, length of the string, and gravity by using dimensional analysis. As will be noted below, this technique, while making simplified assumptions, is extremely powerful and can reveal significant insights, based on algebra alone, on the dependence of simple pendulums on these physical variables.

Dimensional Analysis of the Simple Pendulum.

The period P of the pendulum obviously has dimensions of time, [T]. On the other side of the equation, we need to express the dependence of P on m, l, and g in a general way. To that end, we can write (using the proportionality symbol $\propto$)

$$P \propto m^a l^b g^c. \quad (1)$$

The point here is that the combinations of the dimensions of [M], [L], and [T] on the right-hand side of this relation must be the same as that on the left, namely [T]. The exponents a, b, and c may be of either sign. Since the object of this exercise is dimensional consistency, we replace equation (1) with an equation by introducing a dimensionless constant of proportionality K to yield

$$P = K m^a l^b g^c. \quad (2)$$

Note that K, being dimensionless, is just a number, but the method employed here cannot determine its value. A more detailed study of the dynamics of a simple pendulum reveals that $K = 2\pi$. Do you see any potential flaws in the statement for equation (2)? How do we know that mass, length, and gravity are the only variables that determine the period? We have neglected damping due to friction at the pivot and air resistance, but does it depend on the temperature of the air, or the humidity? As stated earlier, these last two factors are assumed insignificant. What about dependence on the angle of the "swing"? It is clear that to formulate the problem correctly we either have to know something about the answer or have good physical intuition before we are able to answer it! Nevertheless, a certain amount of trial and error ("experimental mathematics") can often pay dividends.

Perhaps an equally important assumption made in the method is that the period should depend on the particular product of power laws stated. Why should it not be a sum or difference, for example, or a transcendental function of some or all those chosen independent variables? The case of a sum or difference is easily recognized as unreasonable because the units of (for example) $m^a + l^b + g^c$ would all be inconsistent, unless several other dimensional constants were to be introduced, defeating the simplicity of the method. The other objection – that of functional dependence – is entirely reasonable, and valid in some cases, as we shall see below. In the relatively rare situations where a more complicated expression is required, it can be seen that, by examining limiting cases, the product formulation is still justified in those instances.

Having pointed out some potential pitfalls of the method, we can now proceed with the simple pendulum example by referring back to equation (2). Expressed in terms of the fundamental dimensions [M], [L], and [T], and recalling that K is dimensionless, we have

$$[T] = [M]^a[L]^b\{[L][T]^{-2}\}^c. \quad (3)$$

We now 'solve' this equation for the exponents $\{a, b, c\}$, but to make this a little easier, we rewrite equation (3) as

$$[M]^0[L]^0[T]^1 = [M]^a[L]^b\{[L][T]^{-2}\}^c. \quad (4)$$

It is clear that we have three equations derived from the three exponents of the dimensions [M], [L], and [T], namely,

$$a = 0; \quad (5a)$$

$$b + c = 0; \quad (5b)$$

$$-2c = 1. \quad (5c)$$

Hence,

$$a = 0, b = 1/2, c = -1/2, \quad (6)$$

and the conclusion follows:

$$P = K(l/g)^{1/2}. \quad (7)$$

Note that the period is *independent* of the mass of the pendulum. Furthermore, the period increases with the pendulum length and decreases when the gravitational acceleration increases (for example, on more massive planets).

Consider a metal pendulum used in a typical grandfather clock. While it is not a simple pendulum, the model does help us see why a clock may lose time during hot summer months. The metal rod, which is *not* inextensible, will lengthen slightly when temperature increases, with a corresponding increase in the period (each 'tic-toc' takes slightly longer time). This decreases the frequency $f (= P^{-1})$ for each oscillation.

To illustrate the pendulum frequency, imagine taking our simple pendulum to the moon, where g is about one-sixth that on earth. For a given value of l, the period, P, would be approximately 2.5 times longer than it would be on earth. Perhaps this is one reason why astronauts used 'hopping' gaits to move on the surface of the moon. Indeed, a very crude model of walking was developed by modeling the human leg as a simple pendulum (Alexander, 1996).

More Examples of Dimensional Analysis.

Einstein's equation: '*The most famous equation in the world.*' A cartoon drawn by Gary Larson depicts an Einstein-like character in front of a blackboard covered with equations that have all been crossed out: $E = mc$, $E = mc^3$, $E = mc^7$, etc (https://medium.com/@_NicT_/on-mornings-f9b4cdd3fc2a). A janitor has just finished tidying his desk, and "Einstein's" eyes bulge as he hears her claim, "*Now that desk looks better. Everything's squared away, yessir, squaaaaaared away*!" (Larson, 2014). Let's do the math on this one, as they say. Energy has dimensions of force multiplied by distance, and as noted above, force has dimensions of mass multiplied by acceleration, so we write the equation

$$E = Km^a c^b \quad (12)$$

in dimensional terms as

$$[M]^1[L]^2[T]^{-2} = [M]^a\{[L][T]^{-1}\}^b, \quad (13)$$

from which we readily see that $a = 1$ and $b = 2$, so Einstein's celebrated mass-energy equivalence formula is indeed given by

$$E = mc^2, \quad (14)$$

(in this case we know that $K = 1$).

How does the speed, v, of gravity waves on the surface of deep water depend on their wavelength λ and gravity g (with surface tension neglected)? Physically the neglect of surface tension is only significant for very short waves (e.g. ripples with wavelengths of several mm or less). If

$$v = K\lambda^a g^b, \quad (15)$$

we have

$$[L][T]^{-1} = [L]^a\{[L][T]^{-2}\}^b, \quad (16)$$

from which $a = b = 1/2$. Thus

$$v = K\sqrt{g\lambda}, \quad (17)$$

i.e. the speed of the wave increases with wavelength. By contrast, for very short waves (ripples), gravity may be neglected but surface tension effects may not. As remarked earlier, the constant K can be determined in general only by a detailed study of the full physical problem, theoretically, experimentally, or in some combination of the two, and will not concern us here. Requiring the governing equation to be dimensionally correct will produce the necessary functional dependence, provided of course that we have accounted for all the relevant physical variables in the statement of the problem.

The reader is encouraged to try the following problems:

Astronomy Problems.

(i) Use the method of dimensional analysis to express the escape speed V of a projectile (e.g. rocket, baseball, etc.) from a spherical planet of radius r and mass m in terms of m, r, and G, the universal gravitational constant (which has dimensions $[L]^3[T]^{-2}[M]^{-1}$).

Answer: $V = K(Gm/r)^{1/2}$.

(ii) Use the method of dimensional analysis to express the radius r of a black hole in terms of its mass m, the gravitational constant G (which has dimensions $[L]^3[T]^{-2}[M]^{-1}$), and the speed of light c.

Answer: $r = K(Gm/c^2)$.

Fluid-dynamical Problems.

(i) Use the method of dimensional analysis to find the pressure P within a soap bubble of radius r and surface tension s.

Answer: $P = Ksr^{-1}$. Smaller bubbles burst more noisily than large ones (i.e.: listen to freshly opened champagne).

(ii) Use the method of dimensional analysis to find the speed v of ripples on the surface of a liquid in terms of the wavelength λ, water density ρ and surface tension s (with gravity neglected).

Answer: $v = K(s/\rho\lambda)^{1/2}$. This time short waves outrun long ones.

Biology Problem.

Use dimensional analysis to find the tail-beat frequency f of a fish in terms of its body length l, muscle stress σ (or force per unit cross sectional area), and the fluid density of water, ρ.

Answer: $f = Kl^{-1}(\sigma/\rho)^{1/2}$. The bigger the fish, the slower the tail beats (watch *Finding Nemo* (again) to verify this). (For more examples, see Pennycuick (1992).)

The Buckingham Pi Theorem.

The method described in this article, dimensional analysis, is based on a technique developed by Lord Rayleigh (1915), sometimes referred to as the Rayleigh method. It can be formalized as an important theorem called *the Buckingham Pi Theorem* (Adam, 2006).

Conclusion.

As someone who, from my early teenage years, wanted to become an astronomer, and yet struggled greatly in my algebra classes, I wish I had been exposed to the beauty and power of the algebraic method described here: dimensional analysis. Using this technique, linear equations in two or three variables are solved with greater ease. These variables are the exponents of three fundamental dimensional units: Mass [M], Length [L], and Time [T]. Understanding dimensional analysis, significant insight can be gained in STEM-related contexts, and the method provides a considerable incentive to engage with "real world" applications.

Additional examples may be found in Adam (2006). A particularly fascinating one concerns the radius r of the shock front from an atomic explosion of energy E in an atmosphere of undisturbed density ρ, expressed as a function of time t, E and ρ. The answer, for those who wish to try this, is $r = K(Et^2/\rho)^{1/5}$.

Note that we can express the entire notation and physical dimensions for any of the models, as shown in the figure below.

Variable	Symbol	Dimensions
Speed	v	$[L][T]^{-1}$
Acceleration	a (or g)	$[L][T]^{-2}$
Period	P	$[T]$
Frequency	f	$[T]^{-1}$
Length/wavelength	l	$[L]$
Mass of pendulum bob	m	$[M]$
Force	F	$[M][L][T]^{-2}$
Muscle stress (force/area or pressure)	σ	$[M][L]^{-1}[T]^{-2}$
Water density	ϱ	$[M][L]^{-3}$
Surface tension (force/length)	s	$[M]][T]^{-2}$
Energy	E	$[M][L]^2[T]^{-2}$

Figure 1. Variables and their physical dimensions.

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