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Teaching Mathematics During the COVID-19 Pandemic!

Every Equation Tells a Story: Waves on Water

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Introduction

Mathematics has been described as the science of patterns, and patterns are all around us – explaining in part why mathematics is so important and so useful! Waves are a fine example of patterns. When I lived in Northern Ireland, I stood sometimes on a high cliff overlooking the North Atlantic Ocean to watch the waves traveling towards the shore. They changed from rolling patterns out in the ocean to rising, curling and breaking patterns as they approached the shore. But are the waves and the water the same thing? Was the water in the breaking waves the same as the water out in the deep ocean? No—that water was at the beach long before the wave arrived. So, *the wave is not the water*—it is the pattern, the shape, the outline of the water surface as it changes in time and space.

Whatever type of wave motion is occurring, there are two things that we can note based on our experience: (i) energy is propagated from points near the source of waves to points which are distant from it, and (ii) the disturbances travel through the medium (whatever that may be) without giving the medium as a whole any permanent displacement.

If we throw a stone into a pond, the ripples spread outwards over the surface carrying energy with them, but if we watch, say, a cork on the surface we see that it, and hence the water, does not move with the waves, but bobs up and down periodically. It is found that whatever the nature of the medium through which the waves are transmitted, whether it be air, a stretched string, a liquid, an electric cable or deep space, the above two properties are common to all types of wave motion, and enable them to be related together. (A caveat: seismic waves *do* cause permanent displacement because they carry so much energy associated with an earthquake).

So what does all this have to do with teaching students mathematics? First, since patterns abound in the world around us, we can readily observe wave motion in water, whether inside or outside the home! Water waves are readily accessible and observable to many students, even if they (the students *or* the waves) are confined to the bathtub or kitchen sink! They can make their own narrow little pond, lake or ocean by putting sand and water in a plastic soda bottle and experimenting with various tipping and sloshing motions (making sure the top is tightly screwed on, of course.) They can add a little 'boat' of some kind to see how it bobs around on the waves. All this can help build scientific and, I argue, mathematical intuition in children. By watching how waves 'react' around protruding plants, rocks and boundaries can help children intuitively appreciate concepts like reflection and even refraction. And this can be very useful in later science classes when students are taught about light and its properties. (A very useful account of 4th graders ideas about light can be found in Chapter 10 of the "How Students Learn" reference).

And when dealing with natural phenomena of all kinds, mathematics and science go hand in hand.

So how can watching waves on *water* help students think about *light*? A useful mental mathematical construct is to ask students to imagine a line (or lines) *perpendicular* to the waves they see rolling in towards the beach, or to the expanding circular waves on a pond or puddle. These lines can be thought of as *rays*, which are a mathematical abstraction we use all the time when thinking of light ("catching some rays"). By thinking about these seemingly different geometrical ideas the student is, in effect being exposed to the complementary descriptive ideas of rays and waves, and establishing in their minds (for the future, perhaps) that things are not necessarily always "either/or" but sometimes "both/and."

The mathematical structure of a wave

Waves do not go on forever of course, but a convenient and very useful mathematical representation of a wave is a sine function, $y = \sin\theta$, for example. This periodic function represents an oscillation of infinite extent. This "wave function" defines the position of a particle in the medium at any position and time as we shall see. There are several basic definitions to introduce in connection with this function: (i) the wave speed (c) – the speed with which is moves to the left or right (in a one-dimensional sense); (ii) the amplitude (a) – the maximum magnitude of the displacement from $y = 0$; (iii) the period (T) – the time for one wave cycle (i.e. from crest to crest or trough to trough) to pass a fixed location; (iv) the frequency (f) – the number of cycles in a unit of time; (v) the wavelength ($\underline{\lambda}$) – the distance between any two points at corresponding positions on successive repetitions in the wave, so (for example) from one crest or trough to the next.

To model a wave using the sine function, consider the ratio of the angle θ and the position x,

$\theta/x = 2\pi/\lambda$, or $\theta = 2\pi x/\lambda$. The sine function has amplitude 1 so multiplying by the amplitude a we can write the function as $y(x) = a\sin(2\pi x/\lambda)$. But we recall from algebra that if $y(x)$ is some function, then $y(x-d)$ is the same function translated in the positive x-direction by a distance d. And in time t, the wave will have traveled a distance ct (in appropriate units of time and distance respectively). Therefore, we may rewrite the wave function as

$$y(x,t) = a\sin\left(\frac{2\pi(x-ct)}{\lambda}\right).$$

In practical situations such as those discussed below a lot of other complicated equations must be solved to be able to write an equation for the speed of waves. Very often they are said to be *dispersive* because the speed c depends on their wavelength as in equation (1) below.

Speed of surface gravity waves

We now examine in detail a fundamental equation describing the speed of waves on the surface of water – an above-mentioned complicated one! For the combined effects of both forces, the speed c of an individual wave crest along a channel of constant depth is (Adam, 2006):

$$c = \lambda f = \left[\left\{\left(\frac{g\lambda}{2\pi}\right)+\left(\frac{2\pi\gamma}{\lambda\rho}\right)\right\}\tanh\left(\frac{2\pi h}{\lambda}\right)\right]^{1/2}, \quad (1)$$

where λ is the wavelength of an individual wave, f is its frequency, ρ is the density of water; h is the depth of the channel and γ is the coefficient of surface tension. The gravitational acceleration is g. Equation (1) also describes a fundamental relationship between the speed, wavelength and frequency of a particular wave, namely $c = \lambda f$. This is the familiar "speed equals distance divided by time" formula in disguise: wavelength is the distance between adjacent crests (or troughs), and frequency is the number of crests (or troughs) that pass a particular point per unit time, so they have dimensions of length and $(\text{time})^{-1}$ respectively. The hyperbolic tangent function also needs to be defined; it is the following combination of exponential functions:

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}. \quad (2)$$

Here, $x = (2\pi h/\lambda) > 0$ since h and λ are both positive. Note that $\lim_{x\to\infty} tanhx = 1$, and $\lim_{x\to 0} tanhx = x$. These limiting cases will be useful in what follows.

Deep Water

(i) Long waves

For our purposes, "deep" here means that the wavelength λ is sufficiently small compared with the depth of the water, i.e. $2,h >> \lambda,$ where the symbol >> means "very much greater than." Of course, this statement (i.e. $h >> 0.16\,\lambda$) is rather vague, and can vary depending on context, but for our purposes even $h \geq \lambda/3$ will suffice, given how rapidly the hyperbolic tangent function approaches one (for example, tanh2 ≈ 0.964 and tanh3 ≈ 0.995). In view of this, the above so-called strict inequalities can be replaced by the approximations $h \geq \lambda/2$ or even $h \geq \lambda/3$. By replacing $\tanh(2\pi h/\lambda)$ with 1, equation (1) now reduces to

$$c = \left[\left(\frac{g\lambda}{2\pi}\right) + \left(\frac{2\pi\gamma}{\lambda\rho}\right)\right]^{1/2}. \quad (3)$$

Now what 'story' does this equation 'tell'? Notice one very important feature of this equation: the channel depth h does not appear. The wave speed is *independent* of the depth; it is the same for any depth channel provided the criterion of 'deep water' is satisfied. In effect, this formula defines the speed for waves that "feel" the effects of gravity and surface tension, but do not "feel" the bottom of the channel (or reservoir, etc.)

But we can take this yet further. For "long" waves i.e. large wavelengths (but still less than $2\pi h$), so that the second term is negligible compared with the first term. This means that the wave motion is dominated by the gravitational force. Then equation (3) reduces to

$$c = \left(\frac{g\lambda}{2\pi}\right)^{1/2}. \quad (4)$$

Since the only variable quantity is λ we see that the speed of individual waves is *proportional to the square root of its wavelength.* Simply put, the longer the wavelength, the faster the wave. Ocean waves are in this category (with the exception of tsunamis which have long wavelengths; see (iii) below).

(ii) Short waves

At the other extreme, we have "short" waves, i.e. the first term is now negligible compared with the second term. Because of this, the assumption of deep water is even more valid than in part (i) above. Now equation (3) takes the form

$$c = \left(\frac{2\pi\gamma}{\rho\lambda}\right)^{1/2}. \quad (5)$$

Now the speed of the wave is *inversely proportional* to the square root of the wavelength. These waves (ripples) are completely dominated by surface tension, and the shorter they are the faster they move. They can be seen fleetingly on a puddle when raindrops fall on them, or even on the gentle slope of longer gravity waves when viewed, say, from a boat on the water.

Shallow water

(i) Long waves

Now let's go to the other extreme from deep water and examine shallow water waves. This means that the depth of water is small compared with the wavelength, i.e. $h << \lambda$. In view of the fact, noted above, that as $\lim_{x\to 0} tanhx = x$ for $x = 2\pi h/\lambda$ (and because the surface tension term can be neglected for large values of λ), formula (1) reduces to the very simple form

$$c \approx \left[\left(\frac{g\lambda}{2\pi}\right)\left(\frac{2\pi h}{\lambda}\right)\right]^{1/2} = \left(gh\right)^{1/2}. \quad (6)$$

These waves *do* "feel" the bottom because of the dependence on the depth h, and the wavelength λ is now absent from the expression. This is an important result: it means that the wave speed is *independent* of wavelength. It follows that all the waves travel with the same speed if the channel depth is constant, and so any complex initial wave configuration may retain an identifiable shape for quite some time afterwards. Tsunamis are generally considered as shallow water waves; with wavelengths of several hundred km they are long compared with typical ocean depths of several km.

(ii) Short waves

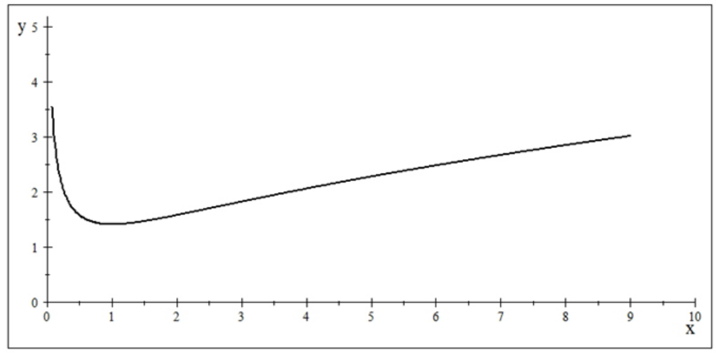
But what about short waves on shallow water? This situation is effectively ruled out because of opposing assumptions: shallow water means waves are long compared with depth, so they will only be considered as short if (using equation (3) in the shallow water limit) $\lambda << 2\pi(\gamma/\rho g)^{1/2}$.

Arithmetic-geometric mean inequality

Let us return to equation (3) for deep-water waves 'driven' by both surface tension and gravity, because there is more of the story to tell. In the extreme cases given by equations (4) and (5) respectively we have seen that the *square* of the speed behaves in a (i) linear and (ii) a rectangular hyperbolic fashion respectively, as functions of wavelength. In the intermediate region, i.e. where the terms $g\lambda/2\pi$ and $2\pi\gamma/\rho\lambda$ are comparable, both forces are also comparable, and the respective graphs of c (λ) must connect. This is illustrated generically in Figure 1 as $c = (\lambda + \lambda^{-1})^{1/2}$.

The arithmetic-geometric mean inequality tells us, in particular, that if $a > 0$ and $b > 0$ then

$$\left(\frac{a+b}{2}\right) \geq \left(ab\right)^{1/2}. \quad (7)$$



Graph of wave speed (c) vs. wavelength λ for a generic choice of $c = (\lambda + \lambda^{-1})^{1/2}$.

Figure 1: Graph of wave speed (c) vs. wavelength λ for a generic choice of $c = (\lambda + \lambda^{-1})^{1/2}$.

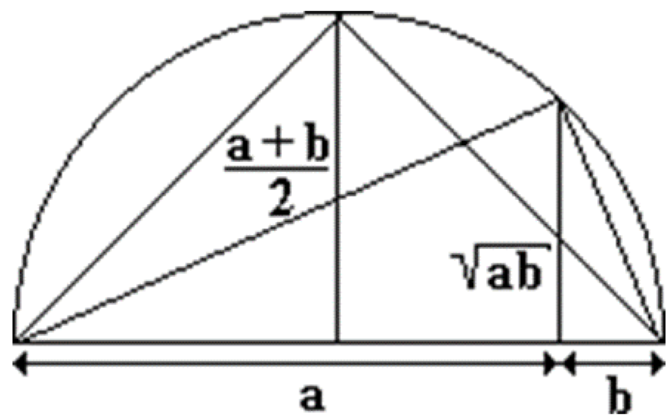


Figure 2: Geometric illustration of the AM-GM inequality.

Equality occurs if and only if $a = b$. This result, which tells us that the arithmetic mean is never less than the geometric mean, is easily established by considering the inequality

$$\left(\sqrt{a} - \sqrt{b}\right)^2 \geq 0.$$

In Figure 2 a geometric representation of this result is shown. The inequality can be generalized to a set of n positive numbers, but we need only two here. Then we can obtain the result we seek by writing equation (3) for brevity as

$$c^2 = \alpha\lambda + \beta\lambda^{-1}. \quad (8)$$

Clearly, if c has a minimum then so does its square. Applying the inequality (7) we see that the sum of the terms in equation (8) is never less than

$$2\left(\alpha\beta\right)^{1/2} = 2\left(g\gamma / \rho\right)^{1/2},$$

and so the minimum speed is

$$c_{\min} = \sqrt{2}\left(g\gamma / \rho\right)^{1/4} \ .$$

Solving equation (8) for λ using this minimum value we find that the value for which this minimum occurs (a double root) is

$$\lambda = \lambda_{cmin} = \left(\frac{\beta}{\alpha}\right)^{1/2} = 2\pi\left(\frac{\gamma}{\rho g}\right)^{1/2} . \quad (9)$$

In principle there is no limit to the *maximum* speed of water waves if their wavelength is small enough. It might be thought that a similar conclusion applies to very long waves as well, but sooner or later the waves in this limit must be considered shallow, and the maximum speed c is then just $(gh)^{1/2}$ as we have seen above. We will put some numbers in here. For water at 20° C, $\gamma \approx 73$ dynes/cm, $\varrho = 1$ gm/cm^3 and $g \approx 981$ cm/s^2, so $\lambda_{cmin} \approx 1.7$ cm, less than one inch. For wavelengths less than or greater than this, the dominant force maintaining the wave motion is respectively surface tension or gravity. The corresponding minimum speed is approximately 23 cm/s. This means that any breeze or gust of wind with speed less than this will not generate any propagating waves, other than a transient disturbance. Wind speeds above this minimum value will in principle generate two sets of waves, with wavelengths on each side of c_{min}, i.e. one set with $\lambda < \lambda_{cmin}$ (ripples) and one set with $\lambda > \lambda_{cmin}$ (gravity waves). Note that these results may also be derived using calculus, and this is summarized in Appendix 1.

Several "equation stories" have been unfolded in this article (and more briefly in Appendix 2) based on a formula for the speed of waves on the surface of bodies of water. Furthermore, implicit connections—some tentative, some more concrete—can been made to teaching mathematics from elementary through middle and high school. In no particular order, these can be summarized more explicitly as follows: (i) - making basic observations and estimates about the speed of waves near the shoreline – are they as fast as a car traveling in heavy traffic? Or in town? Or on a highway? (ii) – elaborating the basic concept of "distance = speed × time" as applied to waves, relating wavelength, speed and period of waves (see equation (1); (iii) – in the small wavelength limit, estimate the speed of waves in puddles – e.g. are they faster than a bee flying from flower to flower? (iv) – use of the exponential function, and relatedly (v) – an introduction to the hyperbolic tangent function and what its graph looks like in several limiting cases; (vi) – algebraic and geometric connections to the arithmetic and geometric means, especially as applied to the smallest possible speed of water waves; (vii) – use of qualitative ideas about speed of waves to explain wave refraction (with implications for the refraction of light); (viii) – a straightforward application of the concept of the derivative to draw conclusions about how circular waves intersect (thus connecting the conic sections circles and hyperbolas); and (ix) – some arithmetical considerations about the speed of tsunamis, speedboats and tides.

As should be apparent from this article, I look for several types of water waves when I make my neighborhood walk in the mornings, especially if it has rained recently. A breath of wind is enough to raise ripples on the surface of puddles, and a solitary raindrop falling from an overhead branch is sufficient to set up a fascinating set of concentric circles, propagating outward smoothly from their center. Then there are longer wind-induced waves frequently visible on the surface of the inlets of the Lafeyette river; rarely is it totally calm, and even then an occasional underwater dweller will break the surface to catch a fly hovering near the surface. Frequently a committee of ducks will launch themselves into the water as I approach them. After the initial splashes have died down, the ducks produce interacting wakes as they head away from me to more suitable gathering place across the water.

So may we all continue to encourage our students, no matter their ages, to enjoy wave hunting!

Appendix 1: *Deriving equation* (9) *using differential calculus*.

The derivative of c^2 with respect to λ is, from equation (8),

$$\frac{dc^2}{d\lambda} = \alpha - \frac{\beta}{\lambda^2} = 0 \text{ when } \lambda = \left(\frac{\beta}{\alpha}\right)^{1/2},$$

on taking the positive root. Since the second derivative is always positive for $\lambda > 0$ it follows that c^2 (and hence c) is a minimum there. That minimum speed is readily found from equation (8).

Appendix 2: Other water-wave related topics for further study.

(i) Wave refraction

It is appropriate at this point to mention wave refraction: as a result of the 'story' above we now have a simple model explaining why ocean waves line up parallel to the beach, even if far out to sea they are approaching it obliquely (see the photograph in Figure 3). Consider the wavelength λ of any particular wave you are observing. Far from the beach, the wave is in deep water, of depth H say. From equation (4) (long waves in deep water), their speed c is proportional to $\sqrt{\lambda}$. For that part of the wave that is closer to the beach, it is in shallow water (of depth h, say, where $H >> h$), so from equation (6) c is proportional to $\sqrt{h}$, which is of course smaller than $\sqrt{\lambda}$. Therefore, that part of the wavefront nearest the beach slows down compared to the part further out, and the whole wavefront "slews" around and lines up parallel to the beach.

Figure 3: Example of waves being refracted parallel to a beach.

(ii) Speedboats

It is of interest to note that speedboats on lakes or harbors or near the beach are often subject to the shallow water speed restriction in equation (6). As this speed is reached, the waves created by the craft just pile up in a big wave ahead of it, and the boat is effectively climbing uphill, making it hard to "power through". In a depth of 6 meters, this critical speed is just under 30 km/hr. Although the speeds are very different, this is similar in some respects to aircraft trying to "break the sound barrier."

(iii) Tides

Interestingly, *tides* are also very shallow water, long period waves. Consider the following very crude (and therefore simplistic) description. As the Earth rotates, the tidal bulges caused by the moon and sun effectively travel around the surface, and at any given moment there are two "high tides" on opposite sides of the Earth, at least if the Earth were a perfectly smooth sphere. The speed of tides in the open ocean is, say 700 km/hr., so every hour there will be a high tide somewhere 700 km farther along the coast. The tidal pattern travels around the globe.

(iv) Ship waves and wakes

The subject of ship waves and wakes has not been addressed here, but interesting discussions of these (and the subject matter in this article) using Google Earth can be found in Aguiar and Souza (2009) and Logiurato (2011). It should also be pointed out that the discussion in this article is based on something called *linear theory*. What this means in principle is that the wave amplitude (crudely, the height) is very small (technically, "infinitesimal"!), so no giant waves or river bores can be described accurately with this theory. In practice, however, it is very useful for many of the types of waves we do see on the surface of oceans, lakes and puddles.

(v) Wave intersections

Having discussed the fundamental equation (1) for surface gravity waves, I share a recent experience. Walking by the water one morning, I noticed a single duck sitting peacefully about thirty yards from me. As it heard my approaching footsteps, it scrambled to "walk on water," flapping its wings to achieve lift as it raced out across the watery runway. Each time its webbed foot touched the water surface, of course, waves were generated. Long before it finally became airborne, a line of these waves started to interfere with each other and produce fascinating intersection patterns. I wish I had brought my camera with me. But it did prompt a related question in my mind. If two pebbles are thrown into a pond one after the other (therefore acting as distinct "point" sources of waves), what is the path of the point(s) of intersection of the waves? However, although these intersections are quite difficult to see in practice, the mathematics below shows that the path is a surprisingly well-known curve.

Figure 4(a): Wave intersections at a moment in time.

These circular waves move outwards in time t with a certain constant speed c. We suppose that the points of intersection of the two circular wave patterns are a distance $r_1(t)$ away from the center of circle 1 and a distance $r_2(t)$ away from the center of circle 2. Since the speed of the waves is constant, then from differential calculus, their speed is the rate of change of radius with respect to time, so

$$c - c = 0 = \frac{dr_1}{dt} = \frac{dr_2}{dt}, \qquad (11)$$

Hence $r_1 - r_2 = \text{constant}$. (12)

But *this* is just the condition that the points of intersection follow a *hyperbolic* path, because a hyperbola is defined as follows: given two distinct points (the foci, here the centers of the circles), a hyperbola is the locus of points such that the difference between the distance to each focus is constant. The resulting intersections for the two sets of waves are shown in Figures 4(a) and 4(b).

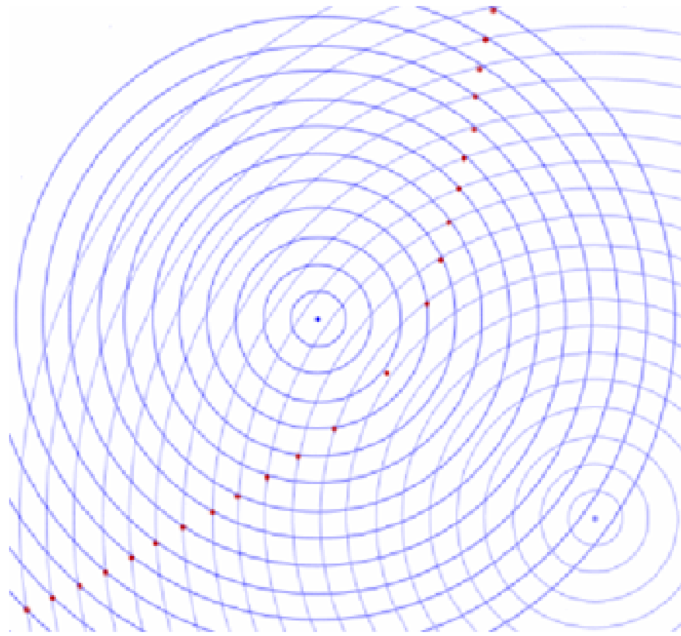


Figure 4(b): Cartoon of circular wave intersections at different times; the dots on the intersecting circles lie on a hyperbola.

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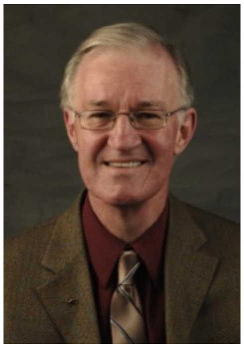

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