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WHAT'S YOUR SPHERICITY INDEX? RATIONALIZING SURFACE AREA AND VOLUME

John A. Adam

Introduction

Virginia Standards of Learning include mathematical content related to the surface area and the volume of various geometric objects. In the seventh grade, "Students... solve problems involving volume and surface area" In the eighth grade, "Proportional reasoning is expounded upon as students solve a variety of problems. Students find the volume and surface area of more complex three-dimensional figures". In high school geometry, "The student... use[s] surface area and volume of three-dimensional objects to solve practical problems" (Virginia Department of Education, 2016). The challenge is to find scenarios that are engaging to students and keep them interested in the context of the mathematics presented to them. In this article, we present real-life situations related to the concepts of ratios, surface area, and volume that are different from the typical content presented in a traditional mathematics textbook. In our experience, students find these problems interesting and engaging. The tasks presented here have the potential to engage students in rigorous thinking about challenging content while using complex, non-algorithmic thinking in order to gain conceptual understanding of the aforementioned mathematical topics.

The Zoological Context

It does not take a zoologist to notice that animals come in all sorts of shapes and sizes. Given the extreme variations in the animal kingdom, how can we gain some understanding of how they relate to their respective environments? One very useful measure is the ratio of the *surface area of an object to its volume* (SA:V). For a cube of side L this is $6/L$, for a sphere of radius R this ratio is $3/R$ (or 6/D, D being the diameter), and for a rectangular box with square bases of side L and length nL this ratio is $[2(1 + 2n)/nL]$.

We consider a *dimensional* ratio, in which its value changes depending on the units of measurement. For example, if $L = 12$ inches, $6/L = ½$ in units of $(\text{inches})^{-1}$, whereas $6/L = 1$ in units of $(\text{feet})^{-1}$, which are inconsistent. In addition, this ratio does not tell us anything about the *shape* of the animal (or object). For example, a thin flat animal or object (like a sting ray or a leaf), with a small volume and a large surface area could have the same SA:V

Figure 1: A Hedgehog

ratio as a sea anemone or a hedgehog (see Figure 1), all quite different shapes. Nevertheless, the latter two examples are apparently much “closer” to being spherical than the former two are.

Note that in every case this ratio is a number divided by a length. This will always be the case because the SA:V ratio has dimensions of $(\text{length})^{-1}$. At this point, we introduce the *sphericity index*, which is a dimensionless ratio that addresses the “shape” issue without regard to the physical size of the object. However, before we introduce it, let us consider a range of “generic” animals, which are all shaped like cubes or spheres when focusing on their exterior shape. That is, we need do is push in their legs, tails and head, pat them around a bit, and we have a cube or a sphere shape. Which we use to make a crude approximation. The area/volume ratio will always be proportional to $(\text{size})^{-1}$ for any type of creature, animate or inanimate. Since any object can be approximated by a collection of cubes or rectangular boxes, these arguments apply in principle to an object of any shape. Initially, the crude estimate of the surface area and volume of any object is made by considering it crudely as a box, and successively, closer approximations can be made by adding more and more smaller boxes to fill in the various gaps. Furthermore, for the simple box models considered in this article, the variable n (a measure of body length) allows for changes in the body size as the animal grows over time.

The SA:V ratio and the sphericity index are essentially complimentary measures; the former, as shown below, gives insight into metabolic rates and requirements, whereas the latter gives insight into its shape, and in particular deviation from the spherical shape. The sphere is optimal in the sense of having the least surface area for a given volume, or equivalently the maximum volume for a given surface area. We shall examine each one in turn.

Surface Area-to-Volume Ratio

What are some of the implications of this simple dimensional ratio? Consider small cubes, where L is small, for example, pygmy shrews, hummingbirds (see Figure 2), beetles, flies or other insects. Roughly speaking, if L is small, $6/L$ is relatively large, and if L is large, $6/L$ is relatively small. Compare a small, cubical shaped, shrew to a large, cubical shaped, elephant. This means, that small animals have a large surface area-to-volume ratio while large animals have a small surface area-to-volume ratio. A consequence of a large ratio is that these animals have a large surface area and therefore, lose heat or gain heat very easily. When the ratio is small, these animals have small surface areas relative to their volume and find it more difficult to lose heat or to gain heat. This is one reason why small warm-blooded animals metabolize, that is, convert food into energy, at such a high rate. They are constantly losing heat to their surroundings and they need to replenish the heat continually when the surroundings are at a lower temperature than their body temperature. Likewise, small cold-blooded creatures are at the mercy of their environment. On the other hand, large animals, like elephants, do not have metabolic rates because they would not be able to lose enough heat to their surroundings through their surface area, which means they would overheat. To compensate for their size, large animals tend to have lower metabolic rates and lower pulse rates. Some animals grow appendages to help them lose heat, for example, the African elephants. They have very large ears that act as efficient radiators. Likewise, some dinosaurs, such as the Dimetrodon may have had sail like appendages on their back for this reason. A simple box model of the long-eared jerboa (Euchoreutes naso) is developed later in this article.

As an exercise, teachers may ask students to consider their own examples created from stiff paper or cardboard to investigate surface areas and volumes by direct measurement. Then, they can calculate surface area to volume ratios.

Although the Sun is not an animal, the same arguments apply. It is a metabolic machine - approximately a sphere with a very, very large radius (about 432,000 miles), so the ratio of area to volume is *exceedingly small*. This means that the effective “metabolic rate” of the Sun is extremely low, but it is enough to keep us functioning on Earth because of its vast absolute surface area:

Figure 2: A Graceful Hummingbird

small energy per unit area multiplied by a very large area = lots of energy.

Strength-to-Weight Ratio

While still focused on dimensional ratios, we can also consider the related *strength-to-weight* ratio. If we take the cross-sectional area of a column or solid bone as a measure of its strength (meaning here the resistance to bending or buckling), then we are on pretty good engineering ground. For a given bone supporting an animal of weight W and size L, its cross-sectional area is proportional to the (size of the animal)^2, i.e. L^2. The weight of the animal is equal to its mass m × the gravitational acceleration g, i.e. $W = mg$, but since mass = volume × density, and volume is proportional to (size)^3 or L^3, it follows that for geometrically similar animals, weight is proportional to L^3.

Hence the strength-to-weight ratio is proportional to $L^2/L^3 = L^{-1}$, i.e. bigger animals appear to be *relatively* less strong than small ones, based on this argument, at least. We can make this statement: if land animals increased in size indefinitely *without* change of shape (i.e. in a geometrically similar fashion), their skeletons would be unable to support them. Their weight would increase faster than the ability of their bones to support their weight. Thus, an animal 3 times the size of another, and geometrically similar would be $3^3 = 27$ times heavier, but only able to support $3^2 = 9$ times the weight of the smaller one. Hence (i) King Kong, as portrayed in the movie, could not exist and (ii) elephants cannot be large mice: their limbs would have to be much thicker relative to their torso than for mice. We now turn to a correspondingly important dimensionless measure.

The Sphericity Index: Description and Definition

This is essentially a dimensionless measure of how *spherical* a three-dimensional shape is, and the fact that it is dimensionless is the key point here. For any closed surface, there must be a dimensionless relationship between its surface area A and volume V of the following form: $A = kV^{2/3}$, (1) where k is a dimensionless constant (i.e. just a number) depending on the shape of the closed surface. From a dimensional perspective, both sides must have dimensions of (length)^2, as already noted, the volume V and surface area A scale respectively as the cube and the square of a linear dimension. It is easy to see that for a cube, $k = 6$.

As an exercise, teachers may want to show that a sphere, where $k = (36\pi)^{1/3}$ which is approximately, 4.836. This leads directly to the sphericity index χ. It is defined as $\chi = (36\pi)^{1/3}\ V^{2/3}/A \approx 4.836\ V^{2/3}/A$, (2). This means, for any sphere shape, the sphericity index, χ, is one. Furthermore, since a sphere has the largest volume-to-surface area ratio for any closed surface, it follows that all other shapes must lie between 0 and 1, $0 < \chi < 1$.

Let us consider two examples, for a cube where the sphericity index is close to one, $\chi \approx 0.806$ and for two “kissing” spheres. That is, these two spheres have a tangential contact (see Figure 3) and a sphericity index that is smaller, $\chi \approx 0.794$. A cube is more spherical in shape than the kissing spheres, but surprisingly, not by much. Let us consider two identical cubes that are in contact with each other at only one corner. Many more such values of χ can be calculated, which makes it fun to do with students. For the rectangular box exercise, discussed in the Introduction, show that the sphericity index is, $\chi \approx 4.836\ n^{2/3}/2(1 + 2n)$.

Figure 3: “Kissing” Spheres

Human Sphericity Index

Many students are interested to calculate their own sphericity index. How close to being spherical are you? I have often given this question as an assignment to my college mathematics students in several classes over many years. However, this activity is appropriate for both middle school students and high-school students. I define the sphericity index and then leave it to them to decide how to estimate their surface area and volume. It is always interesting to see how creative some of them are, but frequently, there is a tendency to over-complicate the problem when students focus on fingers and toes, which has little impact on the final result. On the other hand, if we were to estimate the surface area of a fluffy bath towel or a Christmas tree, the multitude of fibers or pine needles respectively would vastly increase their surface areas compared with a flat sheet (e.g. bath towel) or conical surface (e.g. Christmas tree). Therefore, context is important. Questions like this are designed to help students gain the ability to “model” and “guesstimate” by developing their intuition for what is important, and what can be ignored in mathematical modeling. The question posed here, and the results obtained are invariably enjoyed by the students, and it serves as a great icebreaker for each new class. To estimate human surface area and volume crudely but quickly, without the use of π, as would be the case for a cylinder, we can model the human body as a rectangular box (i.e. parallelepiped) with side lengths *a, b, c*.

Calculating Volume

We may encourage the students to estimate their own surface area and volume in the following way. For example, let us use a typical adult male, who is 6 feet tall. Side $a = 6$, side b = side, where side $c = 1$. Using the volume formula, $V \approx 6 \times 1 \times 1 = 6$ cubic ft. Or, in metric units, since 1 ft. ≈ 0.3 m, it follows the volume is approximately, $V \approx 6 \times (0.3)^3 \approx 0.16\ m^3$. This is probably an overestimate because our legs are not stuck together. For another approach, since most people float in water, the average density of a human is about the same as that of water, or 1 gm/cm^3. This means, one kilogram of you or me occupies about 1000 cm^3, or one liter. A person weighing 170 pounds (i.e. 77 kg) thus has a volume of about 77 liters or roughly $8 \times 10^4 cm^3 = 8 \times 10^4 \times 10^{-6} m^3 = 0.08 m^3$. This is only a factor of two less than the crude upper bound of $0.16m^3$. Therefore, a reasonable estimate is that a typical adult has a volume of about $0.1m^3$. Obviously, middle students and some high school students may need to adjust the measurements appropriately.

Calculating Surface Area

Using the box model as the primary shape, the surface area is $2 \times (6 \times 1 + 6 \times 1 + 1 \times 1) = 26ft^2$, or in metric measurements it is approximately, $26 \times (0.3)^2 \approx 2m^2$. If we were flat like a sheet 2 meters high and 0.5 meters wide, then front and back area gives us the same approximate answer of $2m^2$.

Calculating Sphericity Index

Simplifying, the sphericity index is approximately,

$\chi \approx 4.8\ (V^{2/3} / A) \approx 2.4\ (0.1)^{2/3} \approx 2.4\ (0.22) \approx 0.5(3)$. Doing the same calculation with the same area but the higher volume of 0.16 m^3 gives a corresponding result of 0.7. Therefore, the sphericity index estimate for a typical adult male human is between 0.5 – 0.7. The latter seems a little high, since the sphericity index for a cube is about 0.8. Therefore, I reduce the estimate for an adult male to be in the range 0.5 – 0.6.

Simple Model for a Long-Eared Jerboa
(*Euchoreutes naso*).

This recently-discovered desert animal has ears that are two-thirds as long as its body, and it has the largest ears relative to size in the animal kingdom. Here, we will ignore its long tail and large feet. The Long-Eared Jerboa is typically found in a desert habitat in southern Mongolia and northwest China. Like the African elephant, these giant ears help the jerboa release heat, a vital adaptation in high temperatures. This rodent is about 3 to 3.5 inches from the tip of its nose to the base of its tail, which is twice as long as its body. For this example, we shall ignore the tail and legs and model the animal shape with a rectangular box. We will examine the sphericity index first, and then relate the SA:V approach back to metabolism and the effects of increased surface area relative to volume. We will explore this example by excluding or including the Jerboa's large ears.

(i) No ears. We consider a cuboidal jerboa, a rectangular parallelepiped, with square base of side L and a body length of nL. Its volume, $V = nL^3$, and surface area, $A = 2L^2(1 + 2n)$. It is readily shown from equation (2) that sphericity index is approximately, $\chi \approx 4.836n^{2/3}/[2(1+2n)]$.

(ii) Ears. In this case we append two very thin ears of length $2nL/3$ and height L, but with a volume small enough to be neglected in this simple model. Thus, with two ears there are four surfaces to be added to the previous surface area, so that now $A = 2L^2(1 + 10n/3)$. With ears, the sphericity index is approximately, $\chi \approx 4.836n^{2/3}/[2(1+10n/3)]$.

For the long-eared jerboa the maximum value, approximately 0.573, occurs when $n = 0.6$, which corresponds to the basic body shape that is higher than it is long. This 29% reduction in the sphericity index, χ, is a natural consequence due to a significant increase in the surface area relative to a negligible change in volume. That is, it is less spherical in shape than for case (i).

Both sphericity indices are plotted as a function of n, roughly the length of the jerboa relative to its head size, in Figure 4. Note that χ is maximized for the ear-less jerboa when $n = 1$, (i.e. the animal is a cube). This is not surprising when we recall that the sphericity index for a cube is approximately, $\chi \approx 0.806$, which is the closest to the sphericity index for a sphere, $\chi = 1$.

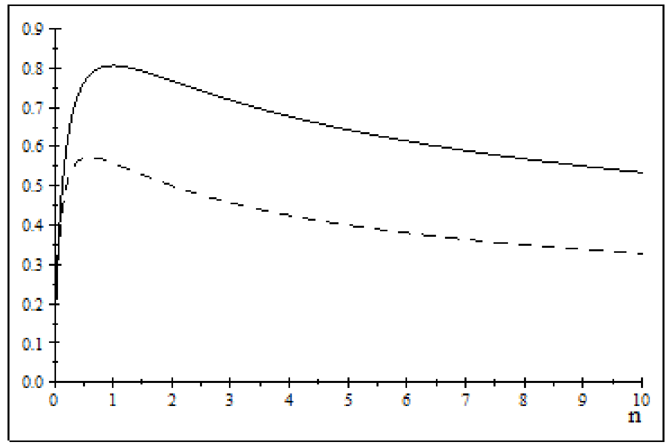

Figure 4: The sphericity index χ(n) for a cuboidal "jerboa" of length nL, both ear-less (solid curve) and with ears (dashed curve).

Back to the SA:V Ratio and Metabolism

As noted earlier, the implications of the dimensional surface area-to-volume ratio can have significant consequences for the metabolic rate of an animal, whereas the dimensionless sphericity index reflects more about the shape of the animal in a general way, in terms of how far it deviates from perfect sphericity. Each represents a different way of understanding aspects of how the animal interacts with its environment. With that in mind, let us re-

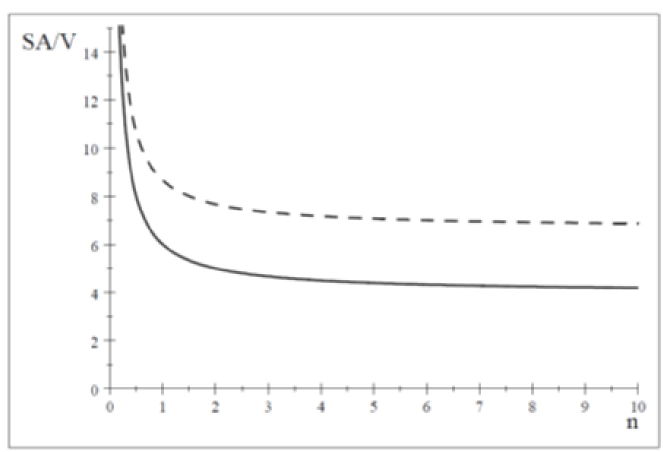

Figure 5: The SA:V ratio for a cuboidal "jerboa" of length nL, both ear-less (solid curve) and with ears (dashed curve).

turn to the two models of the jerboa. For the jerboa with no or very small eared jerboa scenario, the SA:V ratio is $[2(1 + 2n)/nL]$ as noted in the Introduction section. For the long-eared jerboa scenario , the SA:V ratio is $[2(1 + 10n/3)/nL]$. From Figure 5 we note that for all values of n the SA:V ratio for the eared jerboa exceeds that for the earless jerboa. This is due to the increase of surface area afforded by the four surfaces of the ears. Given that these little creatures live in a desert climate, their ears are a valuable mechanism for cooling their bodies, especially since their ears are well-infused with blood vessels.

From 3-D to 2-D: The Circularity Index *C*

The surface area-to-volume ratio for a closed surface has a natural counterpart in two dimensions—the *perimeter-to-area ratio* for a closed bounding curve P. Again, this is a dimensional quantity (with dimensions $(\text{length})^{-1}$), but it is clear, by analogy with equation (1) that $P = \underline{kA}^{1/2}$ for some constant k, depending on the shape of the figure. For a square $k = 4$ and for a circle $k = 2\pi^{1/2}$. Now if we define the *circularity index C* such that $C = kA^{1/2}/P = 1$ for a circle, then it follows that for a square and equilateral triangle respectively $C = \pi^{1/2}/2$ (≈ 0.886) and $C = \pi^{1/2}/3^{3/4}$ (≈ 0.778) respectively.

While such exercises may seem mundane and even purposeless, more sophisticated arguments are relevant to boundaries and areas of legislative districts, urban planning and the socio-political effects of gerrymandering. Lest we go too far astray in this article, consider the simple "district map." The perimeter consists of line segments in units of L, starting at (0,0) and proceeding clockwise as follows: (0,0)→(0,2) →(1,2) →(1,1) →(1,2) →(2,2) →(3,2) →(3,3) →(4,3) →(4,0) →(0,0). As an exercise for the student, show that the area of the district is $A = 8L^2$ and the perimeter is $P = 16L$, so the circularity index is $C = (2\pi)^{1/2}/4 \approx 0.627$. In such a case, both the circularity index and the perimeter-to-area ratio can have implications for the average distribution of populations, their compactness, and the distribution of resources to the region.

Conclusion

The geometric concepts of surface area and volume have been discussed in connection with the surface area-to-volume ratio and the related strength to weight ratio, both applied to species in the animal kingdom. However, these ratios tell us nothing about how close to spherical the actual shape of the animal or object is. In addition, these ratios have dimension of $(\text{length})^{-1}$, and therefore have numerical values dependent on the units of length that are used. A dimensionless ratio is introduced, the, *sphericity index*, that is a useful measure because it is independent of size, but measures proximity to the perfect spherical the shape. A sphere has, by definition, a sphericity index of 1, a cube's sphericity index is approximately 0.806. These concepts lend themselves to discovering more about the geometry of three-dimensional objects and the problem of scale, that is, what happens as objects get bigger (see Langley, 2019 for more information). In two dimensions the corresponding concepts of the perimeter to area ration and the circularity index were discussed as a extension of the sphericity index concept.

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John A. Adam
Mathematics & Statistics,
Old Dominion University
jadam@odu.edu