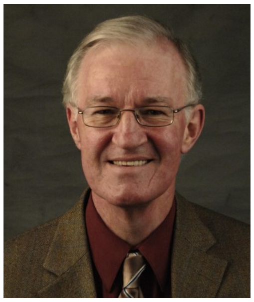

Mathematics in Nature

Section Editor: **John Adam**

In this section we feature articles that examine nature through the lens of mathematics. We use articles that are appropriate for the K-12 classroom. If you wish to share a topic that links mathematics with nature, please respond to the Call for Manuscripts. We are interested in articles that have meaning in the K-12 mathematics classroom.. The following article was contributed by John Adam from Old Dominion University.

Modeling Climate Change

I am extremely grateful to the Editors of VMT for inviting me to be an Editor for the new *Mathematics in Nature* Section. I hope this column provides resources for mathematics teachers to adapt to their own classroom circumstances, and be enjoyable. Initially, I will provide the articles in order to explore topics less familiar with the readership. In the future, this section will contain articles from others who wish to explore the mathematics in nature.

In this issue, my topic focuses on modeling climate change. Climate change is currently considered an "existential threat." Following in the footsteps from a previous author in this journal, Eric Choate who modeled epidemics, I wish to share some features of corresponding details for climate modeling.

The mathematical topics range from applied arithmetic through algebra, solving quadratic equations and geometric series to introductory calculus, differentials. As expected, this topic also includes chemistry and physics.

"The climate is what you expect; the weather is what you get."

This quote is attributed to Mark Twain, and it also appears in the science fiction book, *Time Enough for Love* by Robert Heinlein (1973).

Introduction

In this article I discuss mathematical models of climate change that are within the grasp of middle school and high school mathematics classrooms, in which the mathematics align with many of the Virginia SOLs. These models include important geometric and physical concepts that underlie climate models, without the complexity found in sophisticated models used in practice.

I begin with a brief overview of the history of the climate change, followed by a discussion about the outcomes from burning one gallon of gasoline. The stage is set to describe three increasingly sophisticated models of climate change, which is the theme for this article.

History

It is hard to believe, but the study of what we now refer to as global warming goes back 200 years. In fact, in 1800 the scientist and musician William Herschel discovered the infra-red portion of the electromagnetic spectrum and found it to be hotter than the rest of the visible spectrum. In 1824 Joseph Fourier calculated that a planetary object, the size of Earth, to be cooler than it is, given its distance from the Sun. Therefore, he reasoned, there must be something else apart from the incoming solar radiation that keeps the planet warmer. John Cook on his "Skeptical Science" website states, that "He [Joseph Fourier], suggested that energy coming from the sun in the form of visible and ultraviolet light known in Fourier's time as 'luminous heat,' was able to pass through the atmosphere and heat the planet's surface, but the `non-luminous heat,' now known as infrared radiation, emitted by the Earth's surface, was slowed down on its outward journey back to space." Joseph Fourier recognized that the atmosphere acts as an insulating blanket. There are many other early contributors to the developing study of global warming (see the Skeptical Science website), but we shall concentrate on just two of the more significant ones next.

In 1861, John Tyndall described the way CO_2 inhibits the transmission of infrared radiation. He observed that some gases were transparent to radiated heat whilst others were good absorbers for radiated heat. Water vapor and carbon dioxide, despite being trace gases in the atmosphere, were found to be particularly good absorbers. John Tyndall was particularly interested in the cause of the ice ages and suggested that changes in the amount of CO_2 in the atmosphere could influence the Earth's climate. Later, in 1896 the Swedish scientist, Svante Arrhenius, calculated that Earth's warming would increase when doubling the amount of CO_2 content in the atmosphere. His result was surprisingly accurate by today's standards, with an increase of 5-6°C average warming globally. In his 1908 book, Svante Arrhenius, pointed out that the Earth is about 30 degrees C warmer because of the gases contained in the atmosphere. He claimed CO_2 was an important regulator of the Earth's temperature (See Anderson et al., 2016 for more details). These sections use mathematics aligned to sixth grade SOLs.

Burning One Gallon of Gasoline

In this section, I will show the impact of burning one gallon of gasoline on carbon dioxide, oxygen, and water vapor. Then, we will explore the impact of burning gasoline in the U.S. over a typical year.

How Much Carbon Dioxide is Produced?

When gasoline burns, the carbon and hydrogen in the burning process separate. The hydrogen combines with the oxygen in the air to form water, and the carbon combines with the oxygen in the air to form carbon dioxide. Recall that a carbon atom has an atomic weight of 12, the nucleus contains 6 protons and 6 neutrons. One oxygen atom has an atomic weight of 16, the nucleus contains 8 protons and 8 neutrons. Therefore, the total atomic weight of a molecule of CO_2 is $12 + (2 \times 16) = 44$. When we divide it by the Carbon atomic weight which is $44/12 \approx 3.7$, we find that the atomic weight of the CO_2 molecule is approximately 3.7 times more than the atomic weight of one carbon atom. But, gasoline is about 84% carbon and 16% hydrogen by atomic weight. This means, the carbon in one gallon of gasoline, weighing 6.3 lbs., weighs about 5.5 lb. (0.84×6.3 lbs. = 5.3 lb.). When we multiply this by 3.7, we find that about 19.6 lbs. or approximately 20 lbs. of CO_2 is produced from one gallon of gasoline burned. We can be a little more precise by showing the octane combustion chemical reaction equation:

$2C_8H_{18}+25O_2 \rightarrow 16CO_2+18H_2O$.

Above we showed the molecular weight of CO_2 is 44, and the molecular weight of O_2 is 32. In a similar fashion we calculate the molecular weight of octane (C_8H_{18}), $(8 \times 12) + 18 = 114$. Therefore, from the above reaction equation, every molecule of C_8H_{18} creates 8 molecules of CO_2 with a total molecular weight of $8 \times 44 = 352$. This means, when one gallon of gasoline weighing about 6.3 lb., produces about $6.3 \times (352/114) \approx 19.5$ lbs. of CO_2. The slight numerical differences are due to rounding off some of the numbers used in the calculation.

How Much Oxygen is Burned?

Returning to the reaction equation above, , we see that 25 molecules of oxygen are burned for every 2 molecules of octane. The oxygen molecule has a molecular weight of 32, so the molecular weight of oxygen burned for each octane molecule is $(25/2) \times 32 = 400$ lbs., which means, 6.3 pounds of octane burns $6.3 \times (400/114) \approx 22$ lbs. of oxygen.

How Much Water Vapor is Produced?

I leave it to the readers to show how 9 lbs. of water vapor are produced in the combustion process for one gallon of gasoline burned.

How much CO_2 is put into the atmosphere from US domestic vehicles each year

In this section we continue the conversation using mathematics found in seventh grade SOLs.

We can do an interesting estimation based on some reasonable assumptions. The population of the US is currently about 330 million, which includes children, and not everyone adult owns a car, while other adults own more than one vehicle, while rental car companies have many vehicles Let us estimate that there are 250 million cars on our roads and let estimate that each vehicle averages 12,000 miles per year. Finally, let us estimate that the average fuel consumption of 25 miles per gallon. Given the 20 lbs. of CO_2 produced per gallon from out calculations above, the annual production of carbon dioxide in the US from domestic vehicles is about, $2.5 \times 10^8 \times 1.2 \times 10^4$ (miles)/(vehicle) $\times$ (1/25) (gallons)/(mile)) $\times$ (20 lb.)/(gallon)) $\approx 2.4 \times 10^{12}$ lb. $\approx 10^9$ metric tons, or about one Gigaton (GT) of carbon dioxide per year. We used the conversion in which one metric ton $= 10^3$ kg ≈ 2200 lbs. When we compare this estimate with the graph below that shows the gasoline consumption for motor vehicles in the U.S. as published by the U.S. Energy Information Agency (EIA), we find our estimate is extremely close. We can continue to use estimation, similar to this, to find the amount of oxygen burned in one year in the US and the amount of water vapor produced each year in the U.S. I leave these two exercises to the reader.

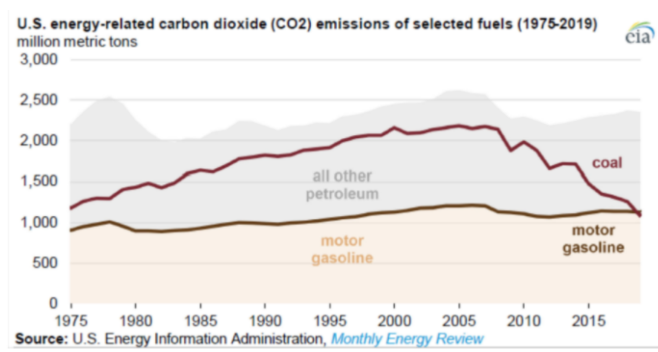

Figure 1: Gasoline consumption for motor vehicles in the U.S.

Additionally, the U.S. EIA estimates that in 2019, the United States emitted 5.1 billion metric tons of energy-related carbon dioxide, while the global emissions of energy-related carbon dioxide totaled 33.1 billion metric tons. As seen from the graph below this may be an underestimate (See. the website, https://ourworldindata.org/co2-emissions, for more information).

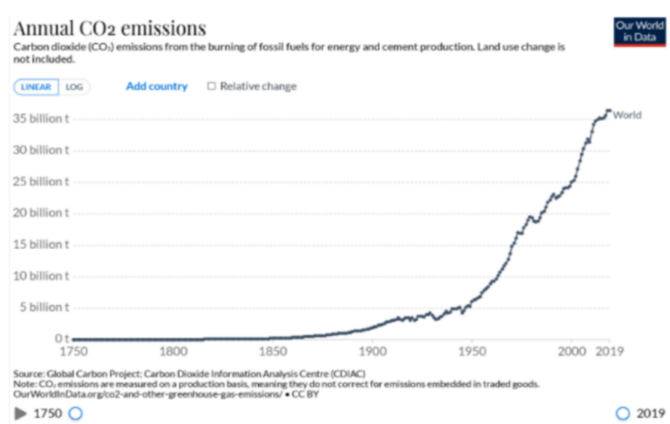

Figure 2: Annual CO2 emissions in the U.S.

Modeling Climate Change: Earth-Sun Systems

We first turn our attention to models that replicate Climate Change, which includes a discussion about the Greenhouse Effect. For any "system" in equilibrium, or balance, this means, the energy going into the system must equal the energy leaving the system. This applies as much to maintaining a constant weight through diet, consumption, and exercise as it does to the Earth receiving radiant energy from the Sun and other sources, whether natural or based on human activity, anthropogenic, and radiating it back out to space. When the energy coming into the climate system balances the energy going out of the climate system the result is a balanced system. This means the averaged global temperature of the Earth will remain constant. When there is an imbalance, one way or the other, this temperature will change. When the input exceeds the output, then the the global averaged temperature will rise. The scientist, Katharine

Hayhoe explains that, it is like the Earth is being covered by an extra blanket – and the Earth sustains a "fever" (2016). We will examine various static models including the Greenhouse effect.

Static Models: Time-Independent

In an excellent article on elementary mathematical models of climate change, authors Daniel Flath *et al.* (2018) discuss what are termed "zero-dimensional energy balance models." In these models, which align with Virginia high school mathematics, there is no spatial or temporal variation, just the temperature of the earth's surface averaged over the whole globe and expressed in terms of some fundamental constants that will be identified below. Although this sounds strange, to use this type of modeling, is very instructive to understand climate at a basic level and to introduce readers to the "art" of mathematical modeling. To that end, following Flath *et al.* (2018), we introduce a sequence of nested zero-order models, which requires we introduce additional physics concepts and terms. We will ask the questions when exploring these models, what went wrong? and what went right?

In a zero-dimensional model the absolute temperature T, in degrees Kelvin, K, is the Earth's surface temperature averaged over the whole globe. The average temperature over large areas of the Earth's surface is a key measure of climate change. To discuss energy balance, we need to equate the energy "in" and the energy "out" using the mathematics model

$E_{in} = E_{out}$, (1)

that neglects mechanisms such as convection and the hydrology cycle which help redistribute energy around the globe without affecting the global energy balance.

Viewed from the Sun, a planet of radius, r_p, presents a circular disk of area, $\pi r^2 p$. We choose the planet to be Earth. The solar flux (Ω), which is the amount of energy per second (i.e., power) per square meter received over this disk, or above the atmosphere, at 1.5×10^8 m from the Sun, is about 1360 W/m², although in reality it does fluctuate a little (±3%) because the Earth's orbit is elliptical. The quantity Ω is often referred to as the solar constant, but it does vary, very slightly over time, and this quantity is unique for each planet. We know the surface area of a sphere of radius r is $S = 4\pi r^2$, so as the Earth rotates between day and night, Ω is distributed over four times the area of the disk, so that the average flux is $\Omega/4$, where flux is the process of flowing in and out.

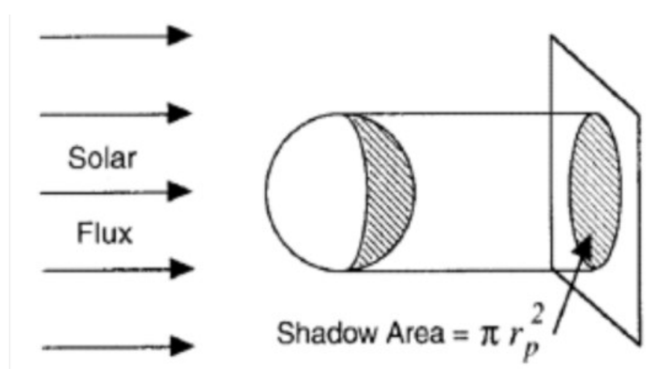

Figure 3: Solar flux diagram

Continuing, it is known that all bodies radiate energy in the form of electromagnetic radiation, and that energy is dependent on the temperature of the body and is proportional to the fourth power of the temperature T. This is the Stefan-Boltzmann law for so-called black-body radiation, which states that the radiant energy output is F, is defined as

$$F = \sigma T^4. \quad (2)$$

The constant of proportionality is $\sigma \approx 5.67 \times 10^{-8}$ W/(m^2K^4). The black body is a perfect absorber of all radiation incident upon it therefore, it is black and it can also emit such radiation. The Earth is not exactly a black body but perhaps surprisingly, this is a reasonable approximation, and this will be modified in Model 3 below. The "energy in" term is given by $\pi r^2 \Omega$, and the "energy out" term is given by SF and the surface area of the Earth multiplied by the radiant energy output per unit area, or $4\pi r^2 \sigma T^4$. Hence, from equations (1) and (2) we have:

Model 1

$$\pi r^2 \Omega = 4\pi r^2 \sigma T^4, \quad (3)$$

or $T = (\Omega/(4\sigma))^{1/4}$. (4)

This is about 278.3 K, or using the conversion from degrees Kelvin to degrees Celsius,

K-273.15 = C, 5.15 degrees Celsius, which is chilly When compared to the current value of about 15 degrees C. In degrees Fahrenheit this model predicts

$$F = (9/5)(K-273.15) + 32, \quad (5)$$

i.e., about 41.3 degrees Fahrenheit.

Model 2: Including *albedo*.

Clouds, snow, and ice are quite efficient at reflecting some of the radiant energy from the Sun back into space, and a measure of the overall reflectivity of the Earth is called its *albedo*, a. The average albedo value is about 0.3. This means that about 30% of the radiant energy received by the Earth is reflected back to space, which most of it is done by clouds. It is important to note that there can be a positive feedback loop associated with this mechanism. That is, the warmer the planet gets, the more ice and snow melt, which means the albedo is reduced, that causes more heat to be absorbed, and the temperature increases, and so forth. This will be addressed in more detail later. In this model equation (3) is modified to become

$$\pi r^2 \Omega(1 - a) = 4\pi r^2 \sigma T^4, \quad (6)$$

so now,

$$T = (\Omega(1-a)/(4\sigma))^{1/4}. \quad (7)$$

This gives a temperature of 254.6 K, or -18.6 degrees C or -1.4 degrees F, which is cold. It appears that when we included more physics, it made the situation much worse. What went wrong? Actually, nothing went wrong, we failed to include Earth's atmosphere. The atmosphere creates the *Greenhouse effect*, which behaves like a blanket around the Earth. The blanket includes various gases, such as carbon dioxide, methane, nitrous oxide, ozone, and water vapor amongst others. The atmosphere increases the surface temperature of our planet, higher than the temperature we calculated in its absence.

The Greenhouse Effect:

The following steps show how the Greenhouse Effect impacts the Earths' temperature when the energy from the sun reaches the Earth and the energy that goes back towards the sun.

1: Solar radiation reaches the Earth's atmosphere - some of this is reflected back into space.

2: The rest of the sun's energy is absorbed by the land and the oceans, which in turn heats the Earth.

3: Heat radiates from the Earth outward towards space.

4: Some of this heat is trapped by greenhouse gases in the atmosphere that keeps the Earth warm to sustain life.

5: Human activities such as burning fossil fuels, agriculture and land clearing are increasing the amount of greenhouse gases released into the atmosphere.

6: This traps extra heat in the atmosphere that causes the Earth's temperature to rise. (See the website for more information about greenhouse effect, https://www.environment.gov.au/climate-change/climate-science-data/climate-science/greenhouse-effect)

Model 3: Black Body Adjustment

In this third model, we adjust our calculations to account for the Earth not being a perfect black body. To do this, we modify the Stephan-Boltzmann law discussed earlier. Informed by Flath *et al.* (2018) examples, we introduce an artificial parameter ε, $0 < \varepsilon < 1$, to modify the Stephan-Boltzmann law because the Earth is not a perfect black body, which is important when examining the greenhouse effect. Equation (6) is modified as follows:

$\pi r^2 \Omega(1 - a) = 4\pi r^2 \varepsilon\sigma T^4$, (8)

so now,

$T = (\Omega(1\text{-}a)/(4\varepsilon\sigma))^{1/4}$. (9)

We can already infer several features from the time-independent model represented by equation (9). For example, the absolute temperature T will increase if the solar flux Ω increases, or if either or both of the albedo, a, and the greenhouse factor, ε, decreases. This is essentially the greenhouse effect, which occurs when $E_{in} = E_{out}$. Conversely, T will decrease if Ω decreases, or a increases (or both), or ε increases. We need to find the value used for ε in order to be consistent with the current global average temperature of 288 K. (See https://ase.tufts.edu/cosmos/view_chapter.asp?id=21&page=1).

From equation (8) we find

$\varepsilon = ((\Omega(1\text{-}a))/(4\sigma T^4))$, (10)

in which

$\varepsilon \approx 0.61$.

So far, the models we used were global. This means we were considering the surface of the Earth and the atmosphere of the Earth as a whole. In reality, there are local variations and this next model addresses this.

Refined Models: Cloud-Earth-Sun System

Previously, we considered an Earth-Sun system, this time the model examines the energy balance for a local Cloud-Earth-Sun system. This more refined model requires we introduce energy balance requirements in terms of energy reflection, transmission, and absorption. The refined models uses infinite geometric series and differential calculus found in advance high school mathematics and college mathematics.

Local radiation balance

When radiant energy encounters an obstacle, that energy may be reflected, transmitted, or absorbed. In general, the energy is a combination of all three mechanisms. For obvious reasons, the proportions of the incoming radiation flux in each of these respective processes can be identified as R, T and A, where R+T+A = 1. Suppose, we examine a local environment in which the energy from the sun encounters two obstacles: 1) a cloud, directly and 2) the surface of the Earth, indirectly. R, T and A will denote the respective proportions for the cloud surface, and R', T' and A' denote the respective proportions for the Earth surface below the cloud. If the incoming solar radiation flux is I_0, then the amount transmitted to the Earth is TI_0 and the amount reflected back into space is RI_0. Of the amount transmitted, a fraction $A'TI_0$ is absorbed while $R'TI_0$ is reflected back outwards and impinges on the base of the cloud. Let us suppose that the coefficients R, T and A are the same for the cloud base and the cloud top, so a proportion $TR'TI_0$ of the incoming flux will be transmitted. This will contribute to the *effective albedo* of the combined cloud-plus-surface system. Continuing this process, as shown in Figure 4, the infinite sequence of terms combines to give the total radiant flux reflected to space as

$F_R = RI_0 + TR'TI_0 + TR'RR'TI_0 + TR'RR'\,RR'TI_0 + \ldots$ (11)

$= RI_0 + TR'[1+RR'+(RR')^2 + \ldots]TI_0$. (12)

In equation (12), the term in square brackets is an infinite geometric series with common ratio RR′ < 1, and the sum is

$$\sum_{n=0}^{\infty}(RR')^n = \frac{1}{1-RR'} . \qquad (13)$$

Therefore, $F_R = RI_0 + (T^2R'I_0)/(1- RR') = I_0(R + (T^2R'I_0))/(1- RR')$, (14)

which means the albedo of the complete system is

$a = F_R / I_0 = R + (T^2R')/(1-RR')$. (15)

Harte (1985) gives an example with $R = 0.5$, $T = 0.4$, $R' = 0.1$ and $T' = 0$. In this case the albedo is

$a = 0.5 + (((0.4)^2(0.1))/(1 - (0.5)(0.1))) \approx 0.52$. (16)

Naturally, the value of *R′,* in particular, varies significantly over different regions of the Earth's surface depending on whether the cloud is above the desert, the ocean, a forest, or an ice field. Students can try various other values to see how sensitive the albedo, *a,* is to changes in *R, T* and *R′*.

Land Use

Next, we explore a situation the local model addresses changes in land use. In the example below, a portion of the land area is deforested and becomes a desert. We examine how this change in local albedo contribute to the global averaged temperature on the Earth. This model is also applicable to other environmental transformation, such as melting glaciers, reforestation, or the increase in the area of large lakes or an ocean. In this modeling process, using differentials is justified for the small changes such as the reflectivity. Basically, given a function *f(x),* say, if *x* changes by an amount δx, then if δx is small enough, the change in *f*, namely $f(x + \delta x) - f(x)$ is approximately δf.

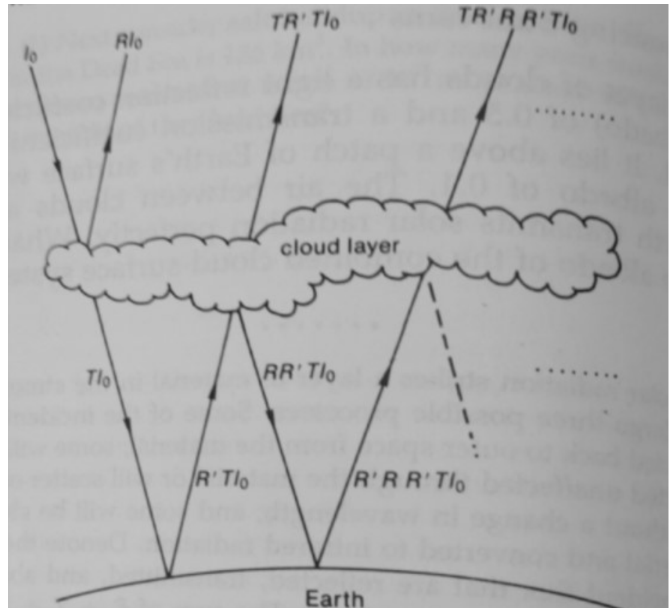

Figure 4: Changes in land use

Another example is shared by Harte (1985). It is as follows: *Suppose that 20% of the land area of Earth is deforested and the area subsequently becomes a desert. By about how much would the Earth's average surface temperature change?*

As noted above, the albedos of the forest versus the desert is different. In fact, the albedo is higher for the desert when compared to the forest. As a result, this will tend to cool the Earth's surface. In addition, deforestation reduces the size of the Earth's "lungs" insofar as trees absorb CO_2. This means the climate-damaging effects of deforestation contributes to the increasing CO_2 in the atmosphere, which almost certainly outweighs the temperature reduction effects.

If a is the albedo of the Earth, let us suppose that this change from forest to desert changes the albedo by an amount $\delta a > 0$. Because of a subtle distinction between the albedo of the Earth (a) and the albedo of the Earth's surface (a_s), changes in the one are not in general quite the same as changes in the other. To understand this relationship, note that

a = ((Solar flux reflected from Earth to space)/ (Solar flux incident on Earth)),

whereas,

a_s = ((Solar flux reflected from Earth's surface to the atmosphere)/(Solar flux incident on Earth's surface)).

Therefore, in practical terms, a is dependent on as. In what follows we use the same notation as in the previous section. Noting that the transmission coefficient $T = 1 - R - A$ we have from equation (15) that

$a = R + (1\text{-}R\text{-}A)^2R'/(1\text{-}RR')$. (17)

It can be shown by similar reasoning to that used in equations (12) - (15) that F_A, the total fraction of the incoming flux that is absorbed in the atmosphere, is given by

$F_A = A[1 + (1\text{-}R\text{-}A)R')/(1\text{-}RR')]$. (18)

We leave it to the reader to find equation (18)

Now, we will provide values to the variable. Let a = 0.3 as before, but what is F_A? Since it is known that about 86 W/m² is absorbed by the atmosphere, F_A = 86/340 ≈ 0.25, and direct measurements reveal that $A \approx 0.23$. If we solve for R' in each of equations (3) and (4) and equate them, we arrive at a quadratic equation in R. Solving this and neglecting a physically unreasonable root for R, we find $R \approx 0.25$ and hence $R' \approx 0.18$. Therefore, from equation (17)

$a \approx 0.25\text{+}(0.52)^2R'/(1\text{-}\ 0.25R') \approx 0.25 + 0.27R'/(1\text{-}0.25R')$. (19)

Now, using calculus we find the rate of change of the Earth's albedo, with respect to the reflectivity at the surface, is given by the following expression:

$da/dR' = 0.27/(1\text{-}\ 0.25R')^2$. (20)

Recall from differential calculus that if the change in R', namely $\delta R'$, is small enough, and the change in a is δa, then we have

$\delta a/\delta R' \approx da/dR'$, (21)

so that

$\delta a \approx (da/dR')\ \delta R' \approx 0.30\ \delta R'$. (22)

Next, we need to find $\delta R'$? Typically, forested land has an albedo of about 0.15 compared with about 0.25 for desert. Recalling that about 29% of the surface of the Earth is land, so 20% of 29% represents about 6% of the total surface area of the Earth. Therefore,

$\delta R' \approx 0.06(0.25\text{-}0.15) = 0.006$, (23)

and so

$\delta a \approx 0.30(0.006) = 0.0018$.

Therefore, the new albedo, a, is approximately 0.3018. For slightly different values of a, Harte (1985) shows that the average surface temperature drops by about 0.3 K.

Again, using differentials, the next section examines how changes in radiation "flux" R (energy per unit time per unit area) is affected by small changes in any or all of the solar flux Ω, albedo a, emissivity ε or temperature T. This is expressed in equation (26), but the same arguments is used to show how changes in any one of these five quantities are dependent on changes in the other four, as developed in equation (27) for example

Recall from differential calculus, if we denote the net flow of radiation per unit area across the "top" of the atmosphere by $R(T)$, which differs from the reflection coefficient used earlier, we may write, using equation (8)

$R = (\Omega/4)(1\text{-}\ a) - \varepsilon\sigma T^4 \equiv R_{in} - R_{out}$ (24)

If the Earth were in perfect energy equilibrium, then $R = 0$. This is essentially another form of the "energy in = energy out" model we used earlier. In this case, we use the zero subscript to denote the equilibrium values, at equilibrium,

$0 = (\Omega_0/4)(1\text{-}\ \mathbf{a_0}) - \varepsilon_0\sigma T_0^4$. (25)

We can now investigate how small changes in the various terms in equation (24) change the value of R, by the amount δR. Using differentials again, we find

$\delta R \approx (1/4)(1 - a_0)\delta\Omega$ - $(1/4)\Omega_0\delta a$ - $4\varepsilon_0\sigma T_0^3\delta T$ – $\sigma T_0^4\delta\varepsilon$. (26)

It is important to note that we are examining a change from one equilibrium state to another. This means, the initial net radiation balance, $R = 0$, is perturbed by an amount δR. This causes the system to evolve, and a new radiation balance is achieved. We can set δR to zero in equation (26). Then we can rearrange the resulting expression, using equation (25), to relate fractional changes in the Earth's surface temperature to the corresponding fractional changes in, respectively, solar output Ω, planetary albedo a, and emissivity ε, namely,

$\delta T/T_0 \approx (1/4)(\delta\Omega/\Omega_0 - \delta a/(1- a_0) - \delta\varepsilon/\varepsilon_0)$. (27)

We examine a special case of this result below by ignoring any changes in the solar flux and the average Earth albedo.

Special Case

In this section we examine the changes in the temperature of the Earth that is induced by an increase in CO_2. The increase in CO_2 uses a crude measure. We begin by setting $\delta\Omega = 0$ and $\delta a = 0$ in equation (27). This means there are no changes in either solar output or albedo. Then, from (27), we find

$\delta T/T_0 \approx - \delta\varepsilon/4\varepsilon_0$. (28)

Equation (28) means that a decrease in emissivity ($\delta\varepsilon < 0$) leads to an increase in average surface temperature ($\delta T > 0$). This is true because such a decrease in emissivity makes it harder for the surface to emit the infrared radiation, which leads to warming. This is important because an increase in CO_2 leads to a decrease in emissivity. In fact, to slightly paraphrase from the book by Randall (2012), It is known from measured optical properties of CO_2 that, for the current climate, a doubling of CO_2 relative to its preindustrial concentration would reduce the outgoing long wave radiation by 4 W/m², so that $\sigma T_0^4\delta\varepsilon \approx$ - 4 W/m². We also know, from satellite observations, that the outgoing long wave radiation $\varepsilon_0\sigma T_0^4$ = 240 W/m². Forming the ratio, we find

- $\delta\varepsilon/\varepsilon_0 = (4/(240)) \approx 0.017$". (29)

This means, that doubling CO_2 creates an approximate 1.7% change to the outgoing long-range radiation. From this we see that using the current globally averaged surface temperature of 288 K, equation (28) implies that doubling CO_2 in the atmosphere would lead to a change in temperature of approximately,

$\delta T \approx$ - $(1/4(\delta\varepsilon/\varepsilon_0)T_0 = (1/4)(0.017)(288) \approx 1.2$ K. (30)

We leave it to the reader to find this temperature change in Celsius and Fahrenheit degrees.

Conclusion

A recent paper by Loeb *et al.* (2021) is very timely, and I draw on this author for my concluding remarks.

"Climate is determined by how much of the sun's energy the Earth absorbs and how much energy Earth sheds through emission of thermal infrared radiation. Their sum determines whether Earth heats up or cools down. Continued increases in concentrations of well-mixed greenhouse gasses in the atmosphere and the long time-scales time required for the ocean, cryosphere, and land to come to thermal equilibrium with those increases result in a net gain of energy, hence warming, on Earth. Most of this excess energy (about 90%) warms the ocean, with the remainder heating the land, melting snow and ice, and warming the atmosphere. Here we compare satellite observations of the net radiant energy absorbed by Earth with a global array of measurements used to determine heating within the ocean, land, and atmosphere, and melting of snow and ice. We show that these two independent approaches yield a decadal increase in the rate of energy uptake by Earth from mid-2005 through mid-2019, which we attribute to decreased reflection of energy back into space by clouds and sea-ice and increases in well-mixed greenhouse gases and water vapor."

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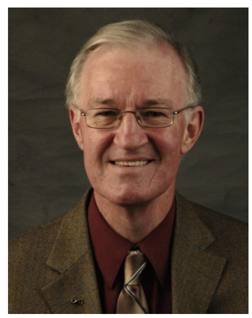

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